Spectral Graph Partitioning

For an unweighted graph $G(V,E)$ with $n$ vertices and $m$ edges, the degree $D$, adjacency $W$ and incidence $A$ matrices lead to the graph Laplacian operator $L = (D-W)$.

An edge separator for a vertex subset $I \subseteq V$ has size:

$$\text{cut}(I,V-I) = \sum_{i \in I, j \notin I} w_{ij}$$

The Ratio-Cut balances the partition by cardinality. For an indicator vector $x \in \mathbb{R}^n$ of $I$, it is approximated by the normalised edge gradients $Ax \in \mathbb{R}^m$.

$$\text{RCut}(I,V) = \frac{\sum_{i \in I, j \notin I} |x[i]|}{|I|} \frac{\sum_{i \notin I, j \in I} |x[j]|}{|V-I|}$$

The Graph $p$-Laplacian

Minimising a vector $p$-norm $|x|_p^p = \sum_{i=1}^n |x[i]|^p$, $p \in (1,2]$ pushes elements of the solution towards discrete values.

We define the scalar function $\varphi_p(x) = |x|^{p-2} \text{sign}(x)$. When applied element-wise to a vector $x \in \mathbb{R}^n$ the inner product returns the $p$-norm.

$$\varphi_p(x) = \sum_{i=1}^n |x[i]|^p$$

The $\varphi_p$ function allows us to re-define the minimisation problem in the $p$-norm and recover the discreteness lost in the Ratio-cut approximation.

$$\min_{x \in \mathbb{R}^n} \frac{|Ax|_p^p}{|x|_p^p} = \min_{x \in \mathbb{R}^n} \frac{1}{|x|_p^p} \sum_{i=1}^n |x[i]|^p \varphi_p(x)$$

This is the Rayleigh quotient for the nonlinear eigenvalue problem that defines the $p$-Laplacian operator $\lambda^p_{\varphi_p}(x) = \lambda_p(x)$.

We exclude the trivial solution with a nonlinear constraint and define the $p$-Laplacian Graph Partitioning [2] problem as follows.

$$\min_{x \in \mathbb{R}^n} \frac{|Ax|_p^p}{|x|_p^p} \quad \text{subject to} \quad \lambda^p_{\varphi_p}(x) = 0$$

Solving the Eigenvalue Problem

The algorithm was adapted to reduce complexity from $O(n^3)$ vertices to $O(m)$ edges for large sparse matrices.

- The nonlinear constraint is handled by feasible projection, using the invertible $\varphi_p$ function.

$$\varphi_p^p(x) = |x|^{p-2} \text{sign}(x)$$

$$\bar{x} = \varphi_p^{-1}(\varphi_p(x) - \frac{1}{\lambda} \lambda^p_{\varphi_p}(x))$$

- Initialising from a 2-Laplacian eigenvector is too costly to compute. Instead a vector is synthesised using METIS [3] software.

- The algorithm is enclosed in an outer loop that decreases $p \rightarrow 1$ improving discreteness and balance during the solve.

Numerical Experiments

The $p$-Laplacian refinement was tested on Delaunay triangulations of $3$-dimensional random point clouds.

- The algorithm ran over five sets of six graphs, each with $1$ to $6$ million vertices ($15$ to $100$ million edges) and calculated the balanced cut metric before and after refinement.

- Results show a consistent improvement of around $4\%$ over the METIS cut, increasing with scale. A similar experiment with $2$-dimensional meshes yields an improvement of around $8\%$.

- Times for the separator refinement are linear in edges $O(m)$ as expected.

When tested on benchmark sparse matrices from the UFL [4] collection, there was a similar improvement of around $4\%$ with more variation due to the different structures encountered.

References


